On log–normal distribution on a comb structure

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We study specific properties of particles transport by exploring an exact solvable model, a so-called comb structure, where diffusive transport of particles leads to subdiffusion. A performance of Lévy – like process enriches this transport phenomenon. It is shown that an inhomogeneous convection flow, as a realization of the Lévy–like process, leads to superdiffusion of particles on the comb structure. A frontier case of superdiffusion that corresponds to the exponentially fast transport is studied and the log–normal distribution is obtained for this case.

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Fractional kinetics attracts much attention and various aspects of these phenomena are reflected in the recent reviews [1, 2, 3, 4, 5]. It should be underlined that while subdiffusion described in the framework of the fractional equations has numerous applications [2, 3, 6, 7, 8], there are only few examples on fractional superdiffusion equations [4, 9, 10] related to Lévy like process in dynamical chaos [4, 9] and consideration of Lévy process for the Langevin equations [10]. As it was mentioned in [5] "the superdiffusion is far from being completely understood". Indeed, it is difficult to imagine the real physical process with the infinite variance jump length during the finite time (Lévy flight). In the Letter we present an example of real physical process which is absolutely analogous to the Lévy flights. In this example we discuss a behavior of observable quantities, namely a position of the center of mass of traveling contaminant and the form of the packet (the tail of a distribution function). The physical mechanism of this Lévy–like effect is an inhomogeneous convection, and the space–time evolution of the contaminant in the presence of the inhomogeneous convection flow is studied. To emphasize this relation between the inhomogeneous convection and the Lévy process, we consider the transport phenomenon on a subdiffusive substrate, *e.g.* on a comb structure.

Our objective is to explain some complications and a possible mechanism of superdiffusion on the comb structure depicted in Fig. 1. It is an analogue of a subdiffusive media where subdiffusion has been already observed [11]. A comb model is known as a toy model for a porous medium used for exploration of low dimensional percolation clusters [12] and electrophoresis process [13]. It should be underlined that conditions or changes that are necessary to perform in the Liouville equation in order to observe superdiffusion in the comb model are important for understanding the nature of this process from the general point of view. We show that a performance of Lévy–like process along, say x–direction, is due to the inhomogeneity of the corresponding x–component of the convection current in the Liouville equation. The diffusion process in such media (modeled by the comb structure) are anomalously slow with the subdiffusive mean squared displacement of the order of $\langle x^2(t)\rangle \sim t^\alpha,\ \alpha < 1$.

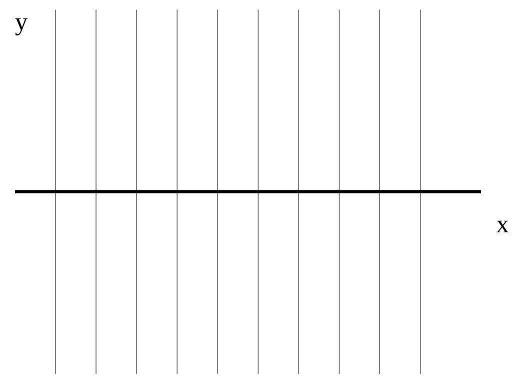

FIG. 1: The comb structure

There are external forces leading to convection. In general case, the velocity of the convection flow is space dependent, *i.e.* convection is inhomogeneous. The question under investigation is how the observable shape of the initial packet changes, when the space–time evolution of the packet corresponds to the convection flow. It should be underlined that this problem arises in a variety of applications starting from transport of external species (pollution) in water flows through porous geological formations [14, 15], problems of diffusion and reactions in porous catalysts [16] and fractal physiology [17, 18, 19]. We find the conditions for the inhomogeneous convection velocity, when the observable packet's shape corresponds to the normal diffusion with $\alpha = 1$ for the mean squared displacement, or even superdiffusion when $\langle x^2(t)\rangle \sim t^\alpha,\ \alpha > 1$. The inhomogeneity will be taken in the form of a power law function $\sim v_s x^s$. In this case, the infinitely long flights take place during a finite fixed time. We will show that the Lévy process takes place when $s > 0$. Our primary objective is the consideration of a frontier case when $s = 1$. The corresponding solution of the Liouville equation for the distribution function of the transport particles is a so–called log–normal distribution.

First, we describe the comb model. A special behavior of the diffusion on the comb structure, depicted in Fig. 1, is that the displacement in the x–direction is possi-

ble only along the structure axis (x-axis, say at $y = 0$). Therefore the diffusion coefficient as well as a mobility is highly inhomogeneous in the y-direction. Namely, the diffusion coefficient is $D_{xx} = \tilde{D}\delta(y)$, while the diffusion coefficient in the y–direction along the teeth is constant $D_{yy} = D$. The Liouville equation

$$\frac{\partial G}{\partial t} + \mathrm{div}\,\mathbf{j} = 0 \tag{1}$$

describes a random walk on this structure with the distribution function $G = G(t, x, y)$, and the current $\mathbf{j} = (-\tilde{D}\delta(y)\frac{\partial G}{\partial x}, -D\frac{\partial G}{\partial y})$. Therefore, it corresponds to the following Fokker–Planck equation

$$\frac{\partial G}{\partial t} - \tilde{D}\delta(y)\frac{\partial^2 G}{\partial x^2} - D\frac{\partial^2 G}{\partial y^2} = 0 \tag{2}$$

with the initial conditions $G(0, x, y) = \delta(x)\delta(y)$ and the boundary conditions on the infinities $G(t, \pm\infty, \pm\infty) = 0$ and the same for the first derivatives respect to x. Performing the Laplace and the Fourier transforms with respect to time and the x–coordinate, correspondingly, one obtains the solution for the subdiffusion along the structure axis [11]

$$G(t, x, 0) = \frac{D^{1/2}}{2\pi\sqrt{\tilde{D}t^3}}\int d\tau \exp\left[-\frac{x^2}{4\tilde{D}\tau} - \frac{D\tau^2}{t}\right] . \tag{3}$$

The total number of particles on the structure axis decreases with time

$$\langle G\rangle = \int_{-\infty}^{\infty} G(t, x, 0)dx = \frac{1}{2\sqrt{\pi Dt}} . \tag{4}$$

Therefore this solution of Eq. (3) describes the subdiffusion when the number of particles is not conserved. In what follows consideration, this point will be bearing in mind [20]. Therefore, this solution corresponds to the subdiffusion with the second moment or the mean squared displacement along the structure axis of the form

$$\langle x^2(t)\rangle = \frac{\langle x^2(t)G(t, x, 0)\rangle}{\langle G(t, x, 0)\rangle} = \tilde{D}\left(\frac{\pi t}{D}\right)^{1/2} . \tag{5}$$

First, let us consider a transport problem on the comb structure due to the constant convection velocity $v_x = v_0\delta(y)$. Here and in what follows one bears in mind that the dimension of $v_s x^s$ is $\tilde{D}x$. The diffusive term in (2) is omitted, for simplicity. Then, we have

$$\frac{\partial G}{\partial t} + v_0\delta(y)\frac{\partial G}{\partial x} - D\frac{\partial^2 G}{\partial y^2} = 0 . \tag{6}$$

Now applying the Laplace and Fourier transform, this equation is solved exactly with the solution in the form

$$G(t, x, 0) = \frac{D^{1/2}x}{\sqrt{\pi t^3 v_0^2}}\exp(-Dx^2/v_0^2 t) , \tag{7}$$

and $G = 0$ for $x < 0$, since the distribution function must be positive. Moreover, the is no transport in the negative direction due to the initial conditions. It corresponds to the normal diffusion with the second moment

$$\langle x^2(t)\rangle = \frac{v_0^2}{D}t . \tag{8}$$

Therefore, to obtain superdiffusion on the comb structure, it is not enough to add a fast process *e.g.* due to the constant convection. In what follows we discuss some mechanism for the superdiffusion, where we suggest an inhomogeneous convection.

A superdiffusive Lévy–like process corresponds to infinite flights during a finite fixed time. Such a process could be a realization of an inhomogeneous current in the x-direction. For example, the following two-dimensional convective current

$$\mathbf{j} = (v_s x^s G(t, x, y)\delta(y), -D\partial G(t, x, y)/\partial y) \tag{9}$$

after substituting in the Liouville equation (1) gives some modification of Eq. (6) in the form

$$\frac{\partial G}{\partial t} + \hat{V}_x G\delta(y) - D\frac{\partial^2 G}{\partial y^2} = 0 , \tag{10}$$

where $s > 0$ and

$$\hat{V}_x G = v_s x^s\frac{\partial G}{\partial x} + s v_s x^{s-1}G . \tag{11}$$

After the Laplace transform with respect to the time, one presents the solution in the form

$$G(p, x, y) = f(p, x)\exp\left[-(p/D)^{1/2}|y|\right] , \tag{12}$$

where we used that

$$\frac{\partial^2}{\partial y^2}\exp\left[-\sqrt{\frac{p}{D}}|y|\right] = \left[\frac{p}{D} - 2\sqrt{\frac{p}{D}}\delta(y)\right]\exp\left[-\sqrt{\frac{p}{D}}|y|\right] .$$

Substituting Eq. (12) in Eq. (10), we obtain then the following equation for $f \equiv f(p, x)$

$$v_s x^s f' + s v_s x^{s-1} f + 2[pD]^{1/2} f = \delta(x) \tag{13}$$

with the following boundary conditions on the infinity $f(p, \infty) = 0$ and $f(p, x) = 0$ for $x < 0$. For $s < 1$, the solution of Eq. (13) has the following form

$$f = \frac{\Theta(x)}{v_s x^s}\exp\left[-\frac{2(Dp)^{1/2}x^{1-s}}{v_s(1-s)}\right] , \tag{14}$$

where $\Theta(x)$ is a step function: $\Theta(x) = 1$ for $x \geq 0$ and it is zero for $x < 0$. It corresponds to some kind of superdiffusion along the structure axis where all moments are determined by the following distribution function

$$G(t, x, 0) = \Theta(x)\frac{D^{\frac{1}{2}}x^{1-2s}}{v_s^2(1-s)\sqrt{\pi t^3}}\exp\left[-\frac{Dx^{2-2s}}{v_s^2(1-s)^2 t}\right] . \tag{15}$$

For example, $\langle x^2(t)\rangle \propto t^{\frac{1}{1-s}}$. When $s=0$, Eq. (15) coincides with Eq. (7). To avoid deficiency at $x=0$ where the solution is infinite, one needs to introduce a diffusion process, by means the second derivative with respect to x. In this case, the equations (10),(11) and (13) describe the space time evolution of the initial profile of particles on the asymptotically large scale $x \gg 1$. Therefore, the solutions (14) and (15) are the asymptotic ones. When $s \geq 1$, it is more convenient to look for a solution on a class of generalized functions. The case with $s>1$ needs a special care that is deserved a separate consideration [21]. When $s=1$, it is a frontier case, with a strong Lévy–like process. The so–called log–normal distribution could be realized here as well.

We consider the case of inhomogeneous transport along the structure axis due to the inhomogeneous drift with $s=1$ and inhomogeneous diffusion of the form $D_{xx}(x,y)=\tilde{d}x^2\delta(y)$. To this end we add the diffusion $-\tilde{d}x^2\delta(y)\partial G/\partial x$ to the x–component of the current in Eq. (9). The Fokker–Planck equation (2) reads now

$$\frac{\partial G}{\partial t}+\hat{V}_x G\delta(y)-D\frac{\partial^2 G}{\partial y^2}=0\,, \tag{16}$$

where diffusion along the structure axis is

$$\hat{V}_x G=-\tilde{d}x^2\frac{\partial^2 G}{\partial x^2}-(2\tilde{d}-v_1)x\frac{\partial G}{\partial x}+vG\,. \tag{17}$$

Again, performing the Laplace transform with respect to the time and presenting the solution of Eqs. (16) and (17) in the form of Eq. (12) we obtain the following equation for $f\equiv f(p,x)$

$$-\tilde{d}x^2 f''-(2\tilde{d}-v_1)xf'+(2[pD]^{1/2}+v_1)f=\delta(x) \tag{18}$$

with the same boundary conditions as in Eq. (2). The following solution is obtained:

$$f(p,x)=\frac{\delta(x)}{2[pD]^{1/2}}+\frac{C_1}{|x|^r}\,, \tag{19}$$

where $r=\frac{1}{2}[(1-b)+\sqrt{(b+1)^2+8(dp)^{1/2}}]$, and we denote $b=v_1/\tilde{d}$ and $d=D/\tilde{d}^2$. The modulus here indicates that the equation is invariant with respect to the inversion $x\to -x$. We also used here that $-x\partial\delta(x)/\partial x=\delta(x)$ and $x^2\partial^2\delta(x)/\partial x^2=2\delta(x)$ on a class of constant probe functions. This δ–function solution for the inhomogeneous equation can also be obtained by the Fourier transform. In this case the specific solution in the Fourier space is $f_k=(2\sqrt{pD})^{-1}$, whose the inverse Fourier transform gives the first term in (19). The constant C_1 could be specified from the initial conditions. It does not carry any important information for the large scale asymptotics, and, without loosing of the generality, it could be equaled to the unity: $C_1=1$.

To perform the inverse Laplace transform it is convenient to use the long time asymptotics $t\to\infty$ ($p\to 0$). In this case $r\approx 1+a\sqrt{p}$ with $a=2d^{1/2}/(b+1)$. It is also convenient to present the second term in (19) in the form of the exponential: $|x|^{-r}=e^{-r\ln|x|}$. Hence the inverse Laplace transform becomes a standard procedure [22], and we obtain the following asymptotic solution:

$$G(t,x,0)=\frac{\delta(x)}{2\sqrt{\pi Dt}}+\frac{a\ln|x|}{2|x|\sqrt{\pi t^3}}\exp\big[-\frac{a^2\ln^2|x|}{4t}\big]\,. \tag{20}$$

We would like to admit here that $|x|\gg 1$ and the logarithmic function is positive. The first term corresponds to the smoothly time–decaying pining distribution, while the second term, which is the most important for the asymptotic transport, corresponds to a some kind of the log-normal distribution. It should be underlined that the asymptotic solution of Eq. (20) is independent of the coefficient $\tilde{d}$, which determines the diffusion along the axis structure. It is natural, since an asymptotic space–time evolution of the initial distribution is determined by the drift component of the current, but not by diffusion. It means that one could consider the convective process analogous to Eqs. (6) and (10) instead of the more general consideration in the framework of Eqs. (16) and (17). In this case, of course, $G(t,x,y)=0$ for $x<0$.

Let us calculate the second moment analogously to Eq. (5). We have then $\langle G\rangle=[1+4(\tilde{d}+v_1)]/\sqrt{\pi Dt}$, and the second moment is

$$\langle x^2 G\rangle=(8\sqrt{t}/a^2)\exp[4t/a^2]\,.$$

Finely, the displacement is

$$\sqrt{\langle x^2(t)\rangle}=\sqrt{\frac{\langle x^2 G\rangle}{\langle G\rangle}}\propto e^{2t/a^2}\,, \tag{21}$$

This exponentially fast spreading is a result of the Lévy–like process introduced inside the system by means of the inhomogeneous displacement current. There is another interesting interpretation of the obtained result. It could be considered as the Poisson distribution respect to the squared logarithm. Namely, if we denote $z=\ln^2|x|$, then we have from (20)

$$P(z)dz=\frac{a}{2\sqrt{\pi t^3}}e^{-a^2z/4t}dz\,. \tag{22}$$

Hence, the average value of z is

$$\langle z\rangle=\big[\int_0^\infty zP(z)dz\big]/\big[\int_0^\infty P(z)dz\big]=4t/a^2\,. \tag{23}$$

It could be understood as the fact that time approaches to the infinity with the same rate as the squared logarithm of the x–coordinate, namely $4t\sim a^2\ln^2|x|$. Therefore, returning to the log–normal part of the distribution function of Eq. (20) we have, approximately, that its asymptotic behavior is

$$G(t\gg 1,|x|\gg 1,y=0)\sim\frac{1}{|x|\ln^2|x|}\,. \tag{24}$$

It is simply to see, that the flux is zero on the infinities. Finely, one obtains that all even moments of x diverge on the large scale asymptotics. The essential difference between the distributions for subdiffusion of Eq. (3) which gives for $|x|, t \gg 1$, that $G(t, |x|, 0) \propto \exp\left[-(x^4/t)^{1/3}\right]$, and the superdiffusion of Eq. (24) is obvious. Therefore, the main mechanism of the transition from subdiffusion in Eq. (5) to superdiffusion in Eq. (21) on the comb structure is the inhomogeneous convection.

In conclusion, we would like to classify a possible response of a substrate with anomalous transport. As it is shown here it depends on the external forcing according to the power rate of the inhomogeneous convection $j_x(x,t) = v_s x^s \delta(y) G(t,x,0)$. As it follows from the solution (15), when $s < 0$ it is subdiffusion [3, 23]. As we shown here, when $s > 0$ it is superdiffusion, Eq, (15). The homogeneous convection with $s = 0$ corresponds formally to the normal diffusion as in Eq. (8), but the effective diffusion coefficient v_0^2/D is determined by the external forcing v_0. The frontier case with $s = 1$ has two features. The first one corresponds to the log–normal distribution of transport particles, where one deals not with a sum of independent random variables but with their multiplication [8]. As the result the exponentially fast transport takes place (see Eq. (21)). The second feature is that the solution satisfies to the natural boundary conditions, when the flux of transporting particles equals zero on the infinities for any time. When $s > 1$, it corresponds to superdiffusion [21], as well.

The final remark is that the relaxation along the structure axis is "inevitable" process due to the item $\tilde{D}\delta(y)\partial G/\partial x$. Therefore, anomalous transport is due to the convection flow only on the asymptotically large scale $x \gg 1$ where the diffusion process could be omitted. This point is studied in detain in [21], where we shown that this approximation corresponds to Liouville–Green asymptotic solution [24].

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